6.26. Remarks:

(a) $G_\delta(f)$ is "periodic" (in the frequency domain) with "period" f_s.

- So, it is sufficient to look at $G_\delta(f)$ between $\pm\frac{f_s}{2}$

(b) The MATLAB script `plotspect` that we have been using to visualize magnitude spectrum also relies on sampled signal. Its frequency domain plot is between $\pm\frac{f_s}{2}$.

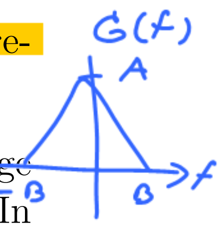
(c) Although this sampling technique is "ideal" because it involves the use of the δ-function. We can extract many useful conclusions.

(d) One can also study the discrete-time Fourier transform (DTFT) to look at the frequency representation of the sampled signal.

6.3 Reconstruction

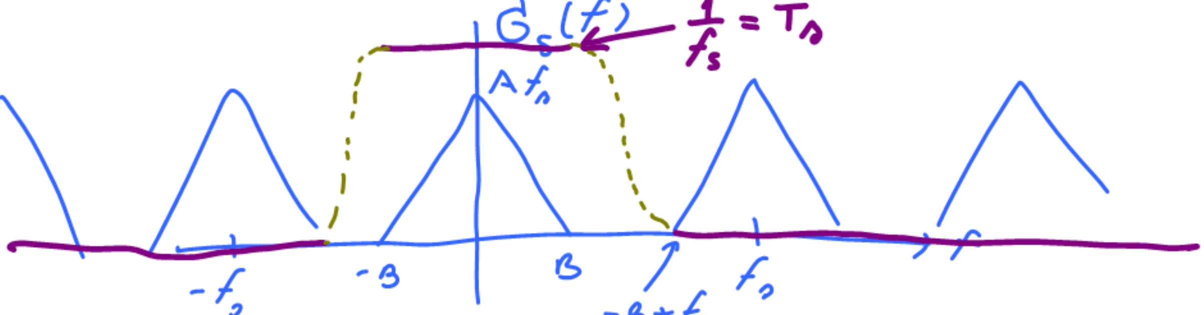
$g(t) \xrightarrow{\text{sampling}} g[n] \xrightarrow{\text{reconstruction}} \hat{g}(t)$

Definition 6.27. Reconstruction (interpolation) is the process of reconstructing a continuous time signal $g(t)$ from its samples.



6.28. From (83), we see that when the sampling frequency f_s is large enough, the replicas of $G(f)$ will not overlap in the frequency domain. In such case, the original $G(f)$ is still intact and we can use a low-pass filter with gain T_s to recover $g(t)$ back from $g_\delta(t)$.

6.29. To prevent aliasing (the corruption of the original signal because its replicas overlaps in the frequency domain), we need



No-overlapping requirement:

$B < -B + f_s$

$f_s > 2B$

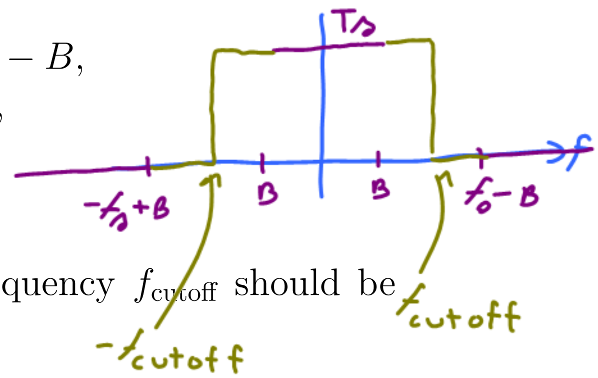
$T_s < \frac{1}{2B}$

Theorem 6.30. A baseband signal g whose spectrum is band-limited to B Hz ($G(f) = 0$ for $|f| > B$) can be reconstructed (interpolated) exactly (without any error) from its sample taken uniformly at a rate (sampling frequency/rate) $f_s > 2B$ Hz (samples per second).[6, p 302]

6.31. Ideal Reconstruction: Continue from 6.28. Assuming that $f_s > 2B$, the low-pass filter that we should use to extract $g(t)$ from $G_\delta(t)$ should be

One important example:

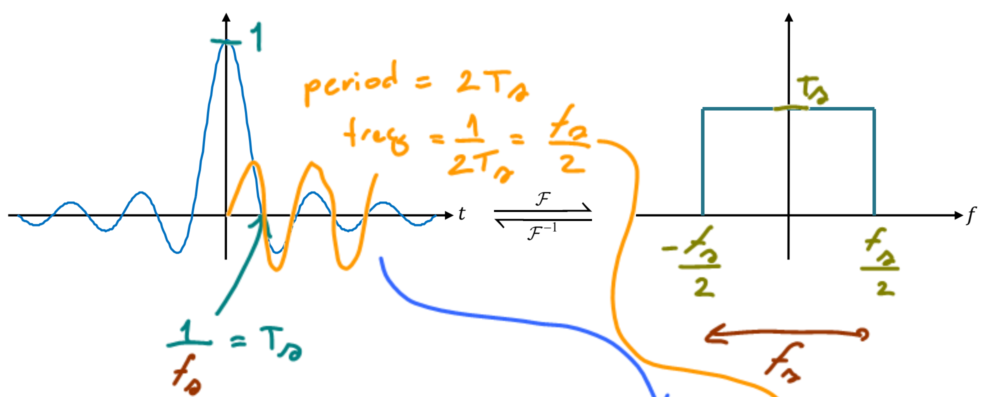
$$H_{LP}(f) = \begin{cases} T_s, & |f| \le B, \\ \text{any}, & B < |f| < f_s - B, \\ 0, & |f| \ge f_s - B, \end{cases}$$



In particular, for "brick-wall" LPF, the cutoff frequency f_{cutoff} should be between B and $f_s - B$.

6.32. Reconstruction Equation: Suppose we use $\frac{f_s}{2}$ as the cutoff frequency for our "brick-wall" LPF in 6.31,

$f_{cutoff} = \frac{f_s}{2}$



The impulse response of the LPF is $h_{LP}(t) = \operatorname{sinc}\left(2\pi\left(\frac{f_s}{2}\right)t\right) = \operatorname{sinc}(\pi f_s t)$. The output of the LPF is

"perceived" signal →

$$\hat{g}(t) = g_\delta(t) * h_{LP}(t) = \left(\sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s)\right) * h_{LP}(t)$$

$$= \sum_{n=-\infty}^{\infty} g[n]h_{LP}(t - nT_s) = \sum_{n=-\infty}^{\infty} g[n]\operatorname{sinc}(\pi f_s(t - nT_s)).$$

When $f_s > 2B$, this output will be exactly the same as $g(t)$:

Reconstruction equation:

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} g[n]\operatorname{sinc}(\pi f_s(t - nT_s))} \tag{84}$$

- This formula allows perfect reconstruction the original continuous-time function from the samples.
- At each sampling instant $t = nT_s$, all sinc functions are zero except one, and that one yields $g(nT_s)$.
- Note that at time t between the sampling instants, $g(t)$ is interpolated by summing the contributions from all the sinc functions.
- The LPF is often called an interpolation filter, and its impulse response is called the interpolation function.

Example 6.33. In Figure 52, a signal $g_r(t)$ is reconstructed from the sampled values $g[n]$ via the reconstruction equation (84).

n=4

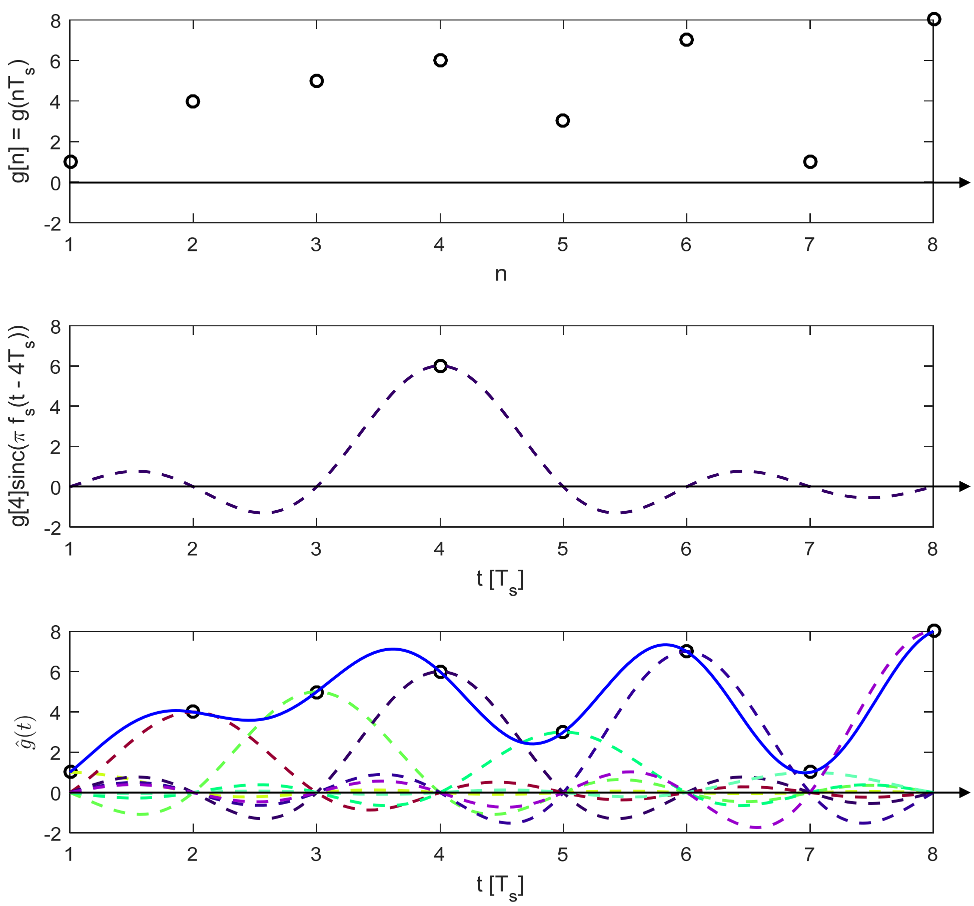


Figure 52: Application of the reconstruction equation

Example 6.34. We now return to the sampling of the cosine function (sinusoid).

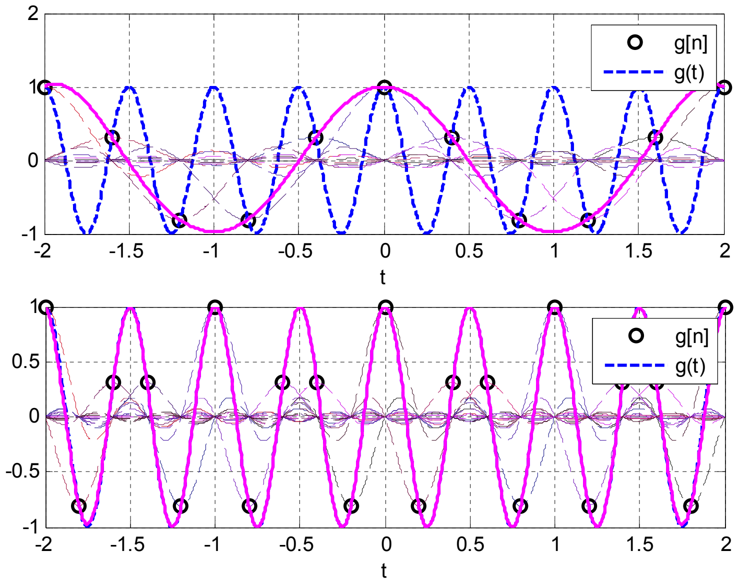


Figure 53: Reconstruction of the signal $g(t) = \cos(2\pi(2)t)$ by its samples $g[n]$. The upper plot uses $T_s = 0.4$. The lower plot uses $T_s = 0.2$.

$f_s = 2.5 < 4$

$f_s = \frac{1}{0.2} = 5 > 4$

Theorem 6.35. Sampling theorem for uniform periodic sampling: If a signal $g(t)$ contains no frequency components for $|f| \geq B$, it is completely described by instantaneous sample values uniformly spaced in time with sampling period $T_s \leq \frac{1}{2B}$. In which case, $g(t)$ can be exactly reconstructed from its samples $(\ldots, g[-2], g[-1], g[0], g[1], g[2], \ldots)$ by the reconstruction equation (84).

6.36. Remarks:

- Need a lot of $g[n]$ for the reconstruction.
- Practical signals are time-limited.
  - Filter the message as much as possible before sampling.

6.37. The possibility of $f_s = 2B$:

- If the spectrum $G(f)$ has no impulse (or its derivatives) at the highest frequency B, then the overlap is still zero as long as the sampling rate is greater than or equal to the Nyquist rate, that is, $f_s \geq 2B$.
- If $G(f)$ contains an impulse at the highest frequency $\pm B$, then $f_s = 2B$ would cause overlap. In such case, the sampling rate f_s must be greater than $2B$ Hz.

Example 6.38. Consider a sinusoid $g(t) = \sin(2\pi(B)t)$. This signal is bandlimited to B Hz, but all its samples are zero when uniformly taken at a rate $f_s = 2B$, and $g(t)$ cannot be recovered from its (Nyquist) samples. Thus, for sinusoids, the condition of $f_s > 2B$ must be satisfied.

Let's check with our formula (83) for $G_\delta(f)$. First, recall that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} = \frac{1}{2j}e^{jx} - \frac{1}{2j}e^{-jx}.$$

Therefore,

$$g(t) = \sin(2\pi(B)t) = \frac{1}{2j}e^{j2\pi(B)t} - \frac{1}{2j}e^{-j2\pi(B)t} = -\frac{1}{2}je^{j2\pi(B)t} + \frac{1}{2}je^{j2\pi(-B)t}$$

and

Note that $G(f)$ is pure imaginary. So, it is more suitable to look at the plot of its imaginary part. (We do not look at its magnitude plot because the information about the sign is lost. We also do not consider the real part because we know that it is 0.)

6.39. The big picture:

- $g(t)$ is a continuous-time signal.
- $g_\delta(t)$ is also a continuous-time signal.
    - However, $g_\delta(t)$ is 0 almost all the time except at nT_s where we have weighted $\delta-$function.
    - We define $g_\delta(t)$ so that we can have an easy way to analyze $g[n]$ below.
      (Another approach is to use DTFT.)
    - It provides an intermediate step that leads to the sampling theorem, the Nyquist sampling rate requirement, and the reconstruction equation.
      It also provides a way to "visualize" aliasing.
- $g[n]$ is a discrete-time signal.
    - This is simply a sequence of numbers.
    - The reconstruction equation says that we can recover $g(t)$ back from $g[n]$ under appropriate condition.
    - So, there is no need to transmit the whole signal $g(t)$. We only need to transmit $g[n]$.